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Twisted cyclic homology of all Podleś quantum spheres

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Abstract

We calculate the twisted Hochschild and cyclic homology of all Podleś quantum spheres relative to diagonal automorphisms. The dimension drop in Hochschild homology is overcome via twisting by the modular automorphism of the canonical $SU_q(2)$-invariant linear functional. Specializing to the standard quantum sphere, we identify the cohomology class of the 2-cocycle discovered by Schmüdgen and Wagner corresponding to the distinguished covariant differential calculus found by Podleś.

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1. Introduction

Twisted cyclic cohomology was discovered by Kustermans, Murphy and Tuset [8], arising naturally from covariant differential calculi over compact quantum groups. They defined a cohomology theory relative to a pair of an algebra $\mathcal{A}$ and an automorphism σ, which on taking $\sigma = \text{id}$ reduces to ordinary cyclic cohomology of $\mathcal{A}$. While it was immediately recognised that twisted cyclic cohomology (and its dual, twisted cyclic homology, the subject of this paper) fits into Connes' general framework of cyclic objects, its relation with differential calculi [16,17] and recent connection with the "dimension drop" phenomenon in Hochschild homology [4–6,19] makes it of independent interest.

Previously [5] we studied the twisted Hochschild and cyclic homology of the quantum $SL(2)$ group. We now extend this work to the Podleś quantum spheres [13,14], which are "quantum homogeneous spaces" for quantum $SL(2)$. The Podleś spheres have been extensively studied, with much work done constructing Dirac operators, spectral triples and the corresponding local index formulae. We mention only [1–3,12] amongst many others. In general, covariant differential calculi over quantum groups do not fit into Connes' formalism of spectral triples [15]. However, in [17] Schmüdgen and Wagner constructed a Dirac operator giving a commutator representation of the distinguished 2-dimensional first order covariant calculus over the Podleś sphere [14]. The associated twisted cyclic 2-cocycle τ was shown to be a nontrivial element of twisted cyclic cohomology. This 2-cocycle does not correspond to the "no

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dimension drop" case — the fact that twisting overcomes the dimension drop in Hochschild homology for the Podleś spheres is the main new result of this paper.

A summary of this paper is as follows. In Section 2 we recall the definitions [5,8] of twisted Hochschild and cyclic homology $HH_*^\sigma(\mathcal{A})$, $HC_*^\sigma(\mathcal{A})$. These "twisted homologies" arise from a cyclic object in the sense of Connes [9], hence all Connes' homological machinery can be applied. Previously we proved that:

Theorem 1.1 *([5]). For arbitrary $\mathcal{A}$ and σ, if σ acts diagonally relative to a set of generators of $\mathcal{A}$ then $HH_n^\sigma(\mathcal{A}) \cong H_n(\mathcal{A}, {}_\sigma\mathcal{A})$ for each n.*

Here ${}_\sigma\mathcal{A}$ is the "σ-twisted" $\mathcal{A}$-bimodule with $\mathcal{A}$ as underlying vector space, and $\mathcal{A}$-bimodule structure

$$a_1 \triangleright x \triangleleft a_2 = \sigma(a_1)xa_2 \quad x, a_1, a_2 \in \mathcal{A}. \tag{1}$$

Since $H_n(\mathcal{A}, {}_\sigma\mathcal{A}) \cong \mathrm{Tor}_n^{\mathcal{A}^e}({}_\sigma\mathcal{A}, \mathcal{A})$ [9] ($\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$), if we have a projective resolution of $\mathcal{A}$ by left $\mathcal{A}^e$-modules, we can in principle compute $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$.

Hochschild and cyclic homology of the Podleś quantum spheres was calculated by Masuda et al. [10], using a free resolution that we rely on in this paper. In Section 3 we recall their definitions. In Section 4 we use this resolution to calculate the Hochschild homologies $H_n(\mathcal{A}, {}_\sigma\mathcal{A})$, which by Theorem 1.1 are isomorphic to the twisted Hochschild homologies $HH_n^\sigma(\mathcal{A})$.

We obtain the following striking result (Theorem 4.6). In the untwisted situation [10] the Hochschild groups $HH_n(\mathcal{A}) = H_n(\mathcal{A}, \mathcal{A})$ vanish for $n \geq 2$, in contrast to the classical situation $q = 1$ (the ordinary 2-sphere) whose Hochschild dimension is 2. This "dimension drop" phenomenon has been seen in many other quantum situations (see [4] for a detailed discussion). However, in the twisted situation, there exist automorphisms σ with $HH_n^\sigma(\mathcal{A}) \neq 0$ for $n = 0, 1, 2$. These automorphisms are precisely the positive powers of the canonical modular automorphism associated to the $SU_q(2)$-invariant linear functional discovered by Noumi and Mimachi [11]. For the standard quantum sphere, which naturally embeds as a subalgebra of quantum $SU(2)$, this modular automorphism coincides with the modular automorphism induced from the Haar state on quantum $SU(2)$. The central role of the modular automorphism in avoiding the dimension drop in Hochschild homology was also seen for quantum $SL(N)$ [5,6]. Similar results have been obtained by Sitarz [19] for quantum hyperplanes.

In Section 5 we calculate twisted cyclic homology as the total homology of Connes' mixed (b, B)-bicomplex arising from the underlying cyclic object. Finally, in Section 6 we apply our results to the standard quantum sphere, showing that the class $[\tau]$ in twisted cyclic cohomology $HC_\sigma^2(\mathcal{A})$ of Schmüdgen and Wagner's twisted cyclic 2-cocycle is proportional to $[Sh_A]$, where S is the periodicity operator and h_A an explicit nontrivial twisted cyclic 0-cocycle.

2. Twisted Hochschild and cyclic homology

We recall the definitions of twisted Hochschild and cyclic homology [5]. Let $\mathcal{A}$ be a unital algebra over a field k (assumed to be of characteristic zero), and σ an automorphism. Define $C_n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}$. For brevity, we will write $a_0 \otimes \cdots \otimes a_n \in \mathcal{A}^{\otimes(n+1)}$ as $(a_0, \ldots, a_n)$. Define the twisted cyclic operator $\lambda_\sigma : C_n(\mathcal{A}) \to C_n(\mathcal{A})$ by $\lambda_\sigma(a_0, \ldots, a_n) = (-1)^n(\sigma(a_n), a_0, \ldots, a_{n-1})$. Hence $\lambda_\sigma^{n+1}(a_0, \ldots, a_n) = (\sigma(a_0), \ldots, \sigma(a_n))$. Now consider the quotient

$$C_n^\sigma(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}/(\mathrm{id} - \lambda_\sigma^{n+1}). \tag{2}$$

If $\sigma = \mathrm{id}$, then $C_n^\sigma(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}$. The twisted Hochschild boundary operator $b_\sigma : C_{n+1}(\mathcal{A}) \to C_n(\mathcal{A})$ is given by

$$b_\sigma(a_0, \ldots, a_{n+1}) = \sum_{j=0}^{n}(-1)^j(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1}(\sigma(a_{n+1})a_0, a_1, \ldots, a_n). \tag{3}$$

We have $b_\sigma^2 = 0$ and $b_\sigma\ \lambda_\sigma^{n+2} = \lambda_\sigma^{n+1}\ b_\sigma$, hence b_σ descends to the quotient, $b_\sigma : C_{n+1}^\sigma(\mathcal{A}) \to C_n^\sigma(\mathcal{A})$. Twisted Hochschild homology $HH_*^\sigma(\mathcal{A})$ is defined as the homology of the complex $\{C_n^\sigma(\mathcal{A}), b_\sigma\}_{n\geq 0}$. Taking $\sigma = \mathrm{id}$ gives $HH_*(\mathcal{A}) = H_*(\mathcal{A}, \mathcal{A})$, the Hochschild homology of $\mathcal{A}$ with coefficients in $\mathcal{A}$.

Now define $C_n^{\sigma,\lambda}(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}/(\mathrm{id} - \lambda_\sigma)$. We have a surjection $C_n^\sigma(\mathcal{A}) \to C_n^{\sigma,\lambda}(\mathcal{A})$. As maps $\mathcal{A}^{\otimes(n+1)} \to \mathcal{A}^{\otimes n}$, we have $b_\sigma(\mathrm{id} - \lambda_\sigma) = (\mathrm{id} - \lambda_\sigma)b'$, where

$$b'(a_0, \dots, a_n) = \sum_{j=0}^{n} (-1)^j (a_0, \dots, a_j a_{j+1}, \dots, a_n). \tag{4}$$

Hence b_σ descends to a map $b_\sigma : C_{n+1}^{\sigma,\lambda}(\mathcal{A}) \to C_n^{\sigma,\lambda}(\mathcal{A})$. Twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ is then defined as the homology of the complex $\{C_n^{\sigma,\lambda}(\mathcal{A}), b_\sigma\}_{n\geq 0}$. Taking $\sigma = \mathrm{id}$ gives back ordinary cyclic homology $HC_*(\mathcal{A})$.

Equivalently, twisted cyclic homology is the total homology of Connes' mixed (b, B)-bicomplex coming from the underlying cyclic object, which we define in Section 5, and use to calculate $HC_*^\sigma(\mathcal{A})$ from $HH_*^\sigma(\mathcal{A})$ for the Podleś spheres.

3. The Podleś quantum spheres

3.1. The coordinate algebras $\mathcal{A}(c, d)$

Let k be a field of characteristic zero, and $q \in k$ nonzero and not a root of unity. For $c, d \in k$, with $c + d \neq 0$, we define the coordinate algebra $\mathcal{A}(c, d)$ of the Podleś quantum 2-sphere $S_q^2(c, d)$ to be the unital k-algebra with generators A, B, B^* satisfying

$$\begin{aligned} &BA = q^2 AB, \quad AB^* = q^2 B^* A \\ &B^*B = cd + (c - d)A - A^2, \quad BB^* = cd + q^2(c - d)A - q^4 A^2. \end{aligned} \tag{5}$$

In the notation of [10], we take $A = \zeta$, $B = Y$, $B^* = -qX$. As algebras, $\mathcal{A}(rc, rd) \cong \mathcal{A}(c, d)$ for any $r \in k$, $r \neq 0$. A Poincaré–Birkhoff–Witt basis for $\mathcal{A}(c, d)$ consists of the monomials

$$\{B^j A^k\}_{j,k\geq 0}, \quad \{B^{*j+1} A^k\}_{j,k\geq 0}. \tag{6}$$

Working over $\mathbb{C}$ (we take $q, c, d \in \mathbb{R}$, with $0 < q < 1$, $0 < c$), for $d > 0$, there is a family of quantum spheres parameterised by $t \in \mathbb{R}$, $t > 0$, with

$$B^*B = t1 + A - A^2, \quad BB^* = t1 + q^2 A - q^4 A^2$$

and also the "equatorial quantum sphere", with $B^*B = 1 - A^2$, $BB^* = 1 - q^4 A^2$. The C*-algebraic completions (with $A^* = A$) were shown by Sheu [18] to all be isomorphic. However, Krähmer proved the underlying algebras are pairwise non-isomorphic [7]. Taking $t = 0$ gives the "standard quantum 2-sphere"

$$B^*B = A - A^2, \quad BB^* = q^2 A - q^4 A^2. \tag{7}$$

Now recall that the coordinate Hopf *-algebra $\mathcal{A}(SU_q(2))$ is the unital *-algebra over $\mathbb{C}$ (algebraically) generated by elements a, c satisfying the relations

$$a^*a + c^*c = 1, \quad aa^* + q^2 c^*c = 1, \quad c^*c = cc^*, \quad ac = qca, \quad ac^* = qc^*a.$$

There is a dual pairing $\langle ., . \rangle$ of $\mathcal{A}(SU_q(2))$ with $U_q(su(2))$, with standard generators $E, F, K^{\pm 1}$ [17], giving left and right actions of $U_q(su(2))$ on $\mathcal{A}(SU_q(2))$:

$$f \triangleright x = \sum \langle f, x_{(2)} \rangle\, x_{(1)}, \quad x \triangleleft f = \sum \langle f, x_{(1)} \rangle\, x_{(2)}. \tag{8}$$

The coordinate *-algebra $\mathcal{A}(S_q^2)$ of the standard Podleś quantum sphere is the *-subalgebra of $\mathcal{A}(SU_q(2))$ invariant under the action of the grouplike element $K \in U_q(su(2))$. Explicitly,

$$a \triangleleft K = q^{-1/2} a, \quad a^* \triangleleft K = q^{1/2} a^*, \quad c \triangleleft K = q^{1/2} c, \quad c^* \triangleleft K = q^{-1/2} c^*.$$

Writing $A = c^*c$, $B = ac$, $B^* = c^*a^*$ gives the relations (5) and (7).

Masuda et al. [10] gave a resolution of $\mathcal{A} = \mathcal{A}(c, d)$,

$$\ldots \to \mathcal{M}_{n+1} \to \mathcal{M}_n \to \ldots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 \to \mathcal{A} \to 0 \tag{9}$$

by free left $\mathcal{A}^e$-modules $\mathcal{M}_n$ ($\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$), with rank$(\mathcal{M}_0) = 1$, rank$(\mathcal{M}_1) = 3$, rank$(\mathcal{M}_n) = 4$ for $n \geq 2$. Adapting their notation, $\mathcal{M}_1$ has a basis $\{e_A, e_B, e_{B^*}\}$, with $d_1 : \mathcal{M}_1 \to \mathcal{M}_0 = \mathcal{A}^e$ given by

$$d_1(e_t) = t \otimes 1 - 1 \otimes t^o, \quad t = A, B, B^*. \tag{10}$$

$\mathcal{M}_2$ has basis $\{e_A \wedge e_B, e_A \wedge e_{B^*}, \vartheta_S^{(1)}, \vartheta_T^{(1)}\}$, with $d_2 : \mathcal{M}_2 \to \mathcal{M}_1$ given by

$$\begin{aligned}
d_2(1_{\mathcal{A}^e} \otimes (e_A \wedge e_{B^*})) &= (A \otimes 1 - 1 \otimes q^2 A^o) \otimes e_{B^*} - (q^2 B^* \otimes 1 - 1 \otimes B^{*o}) \otimes e_A \\
d_2(1_{\mathcal{A}^e} \otimes (e_A \wedge e_B)) &= (q^2 A \otimes 1 - 1 \otimes A^o) \otimes e_B - (B \otimes 1 - 1 \otimes q^2 B^o) \otimes e_A \\
d_2(1_{\mathcal{A}^e} \otimes \vartheta_S^{(1)}) &= -q^{-1}\{B \otimes 1 \otimes e_{B^*} + 1 \otimes B^{*o} \otimes e_B\} - q\{q^2(A \otimes 1 + 1 \otimes A^o) - (c-d)\} \otimes e_A \\
d_2(1_{\mathcal{A}^e} \otimes \vartheta_T^{(1)}) &= -q^{-1}\{1 \otimes B^o \otimes e_{B^*} + B^* \otimes 1 \otimes e_B\} - q^{-1}\{(A \otimes 1 + 1 \otimes A^o) - (c-d)\} \otimes e_A.
\end{aligned} \tag{11}$$

$\mathcal{M}_3$ has basis $\{e_A \wedge \vartheta_S^{(1)}, e_A \wedge \vartheta_T^{(1)}, e_{B^*} \wedge \vartheta_S^{(1)}, e_B \wedge \vartheta_T^{(1)}\}$, with $d_3 : \mathcal{M}_3 \to \mathcal{M}_2$

$$\begin{aligned}
&d_3(1_{\mathcal{A}^e} \otimes (e_A \wedge \vartheta_S^{(1)})) = (A \otimes 1 - 1 \otimes A^o) \otimes \vartheta_S^{(1)} + q^{-3}\{B \otimes 1 \otimes (e_A \wedge e_{B^*}) + 1 \otimes B^{*o} \otimes (e_A \wedge e_B)\} \\
&d_3(1_{\mathcal{A}^e} \otimes (e_A \wedge \vartheta_T^{(1)})) = (A \otimes 1 - 1 \otimes A^o) \otimes \vartheta_T^{(1)} + q^{-1}\{1 \otimes B^o \otimes (e_A \wedge e_{B^*}) + B^* \otimes 1 \otimes (e_A \wedge e_B)\} \\
&d_3(1_{\mathcal{A}^e} \otimes (e_{B^*} \wedge \vartheta_S^{(1)})) \\
&\quad = B^* \otimes 1 \otimes \vartheta_S^{(1)} - 1 \otimes B^{*o} \otimes \vartheta_T^{(1)} - q^{-1}\{(A \otimes 1 + 1 \otimes q^2 A^o) - (c-d)\} \otimes (e_A \wedge e_{B^*}) \\
&d_3(1_{\mathcal{A}^e} \otimes (e_B \wedge \vartheta_T^{(1)})) \\
&\quad = B \otimes 1 \otimes \vartheta_T^{(1)} - 1 \otimes B^o \otimes \vartheta_S^{(1)} - q^{-1}\{(q^2 A \otimes 1 + 1 \otimes A^o) - (c-d)\} \otimes (e_A \wedge e_B).
\end{aligned} \tag{12}$$

$\mathcal{M}_4$ has basis $\{e_A \wedge e_{B^*} \wedge \vartheta_S^{(1)}, e_A \wedge e_B \wedge \vartheta_T^{(1)}, \vartheta_S^{(2)}, \vartheta_T^{(2)}\}$, with $d_4 : \mathcal{M}_4 \to \mathcal{M}_3$

$$\begin{aligned}
&d_4(1_{\mathcal{A}^e} \otimes (e_A \wedge e_{B^*} \wedge \vartheta_S^{(1)})) \\
&\quad = (A \otimes 1 - 1 \otimes q^2 A^o) \otimes (e_{B^*} \wedge \vartheta_S^{(1)}) - q^2 B^* \otimes 1 \otimes (e_A \wedge \vartheta_S^{(1)}) + 1 \otimes B^{*o} \otimes (e_A \wedge \vartheta_T^{(1)}) \\
&d_4(1_{\mathcal{A}^e} \otimes (e_A \wedge e_B \wedge \vartheta_T^{(1)}) \\
&\quad = (q^2 A \otimes 1 - 1 \otimes A^o) \otimes (e_B \wedge \vartheta_T^{(1)}) - B \otimes 1 \otimes (e_A \wedge \vartheta_T^{(1)}) + 1 \otimes q^2 B^o \otimes (e_A \wedge \vartheta_S^{(1)}) \\
&d_4(1_{\mathcal{A}^e} \otimes \vartheta_S^{(2)}) = -q^{-1} B \otimes 1 \otimes (e_{B^*} \wedge \vartheta_S^{(1)})) - q^{-1} \otimes B^{*o} \otimes (e_B \wedge \vartheta_T^{(1)}) \\
&\qquad\qquad - q[q^2(A \otimes 1 + 1 \otimes A^o) - (c-d)] \otimes (e_A \wedge \vartheta_S^{(1)})) \\
&d_4(1_{\mathcal{A}^e} \otimes \vartheta_T^{(2)}) = -q^{-1} \otimes B^o \otimes (e_{B^*} \wedge \vartheta_S^{(1)}) - q^{-1} B^* \otimes 1 \otimes (e_B \wedge \vartheta_T^{(1)}) \\
&\qquad\qquad - q^{-1}((A \otimes 1 + 1 \otimes A^o) - (c-d)) \otimes (e_A \wedge \vartheta_T^{(1)}).
\end{aligned}$$

We refer the reader to [10] for the $\mathcal{M}_n$ and d_n for $n \geq 5$. In Section 4 we use this resolution to calculate the Hochschild homology $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$ of $\mathcal{A} = \mathcal{A}(c, d)$ with coefficients in the twisted $\mathcal{A}$-bimodule ${}_\sigma\mathcal{A}$ defined in (1).

3.2. Comparison of the M–N–W and bar resolutions

We wish to identify generators of $H_*(\mathcal{A}, {}_\sigma\mathcal{A})$, found as elements of the modules $\mathcal{M}_n$, with Hochschild cycles realised as elements of $\mathcal{A}^{\otimes n}$. Recall [9] the bar resolution, with differential b' given by (4)

$$\ldots \to \mathcal{A}^{\otimes(n+2)} \to^{b'} \mathcal{A}^{\otimes(n+1)} \to \ldots \to \mathcal{A}^{\otimes 2} \to^{b'} \mathcal{A} \to 0$$

which is a projective resolution of $\mathcal{A}$ as a left $\mathcal{A}^e$-module. Each $\mathcal{A}^{\otimes(n+1)}$ is a left $\mathcal{A}^e$-module via $(x \otimes y^o)(a_0, a_1, \ldots, a_n) = (xa_0, a_1, \ldots, a_n y)$. The comparison theorem (see, for example [20], Theorem 2.2.6) says that given a projective resolution $\ldots \to \mathcal{M}_1 \to^{d_1} \mathcal{M}_0 \to^{d_0} \mathcal{B} \to 0$ of a left $\mathcal{A}$-module $\mathcal{B}$, and a map $f : \mathcal{B} \to \mathcal{C}$, then for every resolution $\ldots \to \mathcal{N}_1 \to \mathcal{N}_0 \to^{\eta} \mathcal{C} \to 0$ there is a chain map $\{f_i : \mathcal{M}_i \to \mathcal{N}_i\}_{i\geq 0}$, unique up to chain homotopy equivalence, lifting f in the sense that $\eta \circ f_0 = f \circ d_0$. In our situation, taking $\mathcal{B} = \mathcal{C} = \mathcal{A}$ and $f = \mathrm{id}$, maps $f_i : \mathcal{M}_i \to \mathcal{A}^{\otimes(i+2)}$ giving a commutative diagram

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathcal{M}_3 & \xrightarrow{d_3} & \mathcal{M}_2 & \xrightarrow{d_2} & \mathcal{M}_1 & \xrightarrow{d_1} & \mathcal{M}_0 & \xrightarrow{d_0} & \mathcal{A} & \longrightarrow & 0 \\
 & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \cong\downarrow & & \\
\ldots & \longrightarrow & \mathcal{A}^{\otimes 5} & \xrightarrow{b'} & \mathcal{A}^{\otimes 4} & \xrightarrow{b'} & \mathcal{A}^{\otimes 3} & \xrightarrow{b'} & \mathcal{A}^{\otimes 2} & \xrightarrow{b'} & \mathcal{A} & \longrightarrow & 0
\end{array}$$

are given by, in the notation of the previous section:

$$\begin{aligned}
&f_0(a_1 \otimes a_2^o) = (a_1, a_2), \quad f_1(e_t) = (1, t, 1), \quad t = A,\ B,\ B^* \\
&f_2(e_A \wedge e_{B^*}) = (1, A, B^*, 1) - q^2(1, B^*, A, 1) \\
&f_2(e_A \wedge e_B) = q^2(1, A, B, 1) - (1, B, A, 1) \\
&f_2(\vartheta_S^{(1)}) = -q^{-1}(1, B, B^*, 1) - q^3(1, A, A, 1) - q^{-1}cd(1, 1, 1, 1) \\
&f_2(\vartheta_T^{(1)}) = -q^{-1}(1, B^*, B, 1) - q^{-1}(1, A, A, 1) - q^{-1}cd(1, 1, 1, 1).
\end{aligned} \tag{13}$$

Higher f_i can be found inductively: the above is as much as we will need in the following. Applying ${}_\sigma\mathcal{A} \otimes_{\mathcal{A}^e} -$ to both resolutions allows us to identify generators of homology found from the M–N–W resolution with explicit Hochschild cycles.

3.3. Automorphisms of $\mathcal{A}(c, d)$

We consider automorphisms acting diagonally with respect to the generators A, B, B^*. For $c \neq d$, every such automorphism is of the form

$$\sigma_\lambda(B) = \lambda B, \quad \sigma_\lambda(A) = A, \quad \sigma_\lambda(B^*) = \lambda^{-1} B^* \tag{14}$$

some $\lambda \in k$, $\lambda \neq 0$. If $c = d$, there is a second family of diagonal automorphisms

$$\tau_\lambda(B) = \lambda B, \quad \tau_\lambda(A) = -A, \quad \tau_\lambda(B^*) = \lambda^{-1} B^*. \tag{15}$$

We conjecture that every automorphism of $\mathcal{A}(c, d)$ is of this form. It follows from Theorem 1.1 that:

Lemma 3.1. *$HH_n^\sigma(\mathcal{A}) \cong H_n(\mathcal{A}, {}_\sigma\mathcal{A})$ for all $n \geq 0$ and every $\sigma = \sigma_\lambda, \tau_\lambda$.*

Working over $\mathbb{C}$, Noumi and Mimachi [11] proved the existence of a unique linear functional $h : \mathcal{A}(c, d) \to \mathbb{C}$ invariant under the left coaction of quantum $SU(2)$, and satisfying $h(1) = 1$. On monomials this is given by

$$h(B^{m+1}A^n) = 0 = h((B^*)^{m+1}A^n), \quad h(A^n) = \frac{f(0)}{f(n)}\left(\frac{c^{n+1} - (-d)^{n+1}}{c + d}\right) \tag{16}$$

where $f(n) = q^{-2} - q^{2n}$. h is a twisted cyclic 0-cocycle. Borrowing terminology used for quantum $SU(2)$, the unique automorphism σ_{mod} satisfying $h(xy) = h(y\sigma_{\mathrm{mod}}(x))$ is called the modular automorphism (so h is a $\sigma_{\mathrm{mod}}^{-1}$-twisted 0-cocycle). Concretely,

$$\sigma_{\mathrm{mod}} : \quad A \mapsto A, \quad B \mapsto q^{-2}B, \quad B^* \mapsto q^2 B^*. \tag{17}$$

Obviously σ_{mod} is well-defined over any field, not just $\mathbb{C}$. As previously seen, the standard Podleś quantum sphere embeds as a subalgebra of quantum $SU(2)$, and the modular automorphism associated to the Haar state on quantum $SU(2)$ restricts to an automorphism of the standard Podleś sphere coinciding with (17).

4. Twisted Hochschild homology

We calculate the Hochschild homologies $H_n(\mathcal{A}, {}_\sigma\mathcal{A})$ of $\mathcal{A} = \mathcal{A}(c, d)$ for all automorphisms $\sigma = \sigma_\lambda, \tau_\lambda$ using the Masuda–Nakagami–Watanabe resolution (9). By Lemma 3.1 we can identify these with $HH_n^\sigma(\mathcal{A})$. The case $\sigma = \mathrm{id}$ was already treated in [10]. In each case we exhibit explicit generators.

4.1. $HH_0^\sigma(\mathcal{A})$

Let $\sigma_\lambda, \tau_\lambda$ be the automorphisms of $\mathcal{A}(c, d)$ given by (14) and (15).

Proposition 4.1. *For arbitrary c and d (with $c + d \neq 0$) and $\sigma = \sigma_\lambda$ we have:*

1. *For $\lambda = 1$ ($\sigma = \mathrm{id}$), $HH_0^\sigma(\mathcal{A})$ is countably infinite dimensional.*
2. *For $\lambda \neq 1$, $HH_0^\sigma(\mathcal{A}) \cong k^2$.*

For $c = d$, and $\sigma = \tau_\lambda$ we have:

1. *For $\lambda = 1$, $HH_0^\sigma(\mathcal{A})$ is countably infinite dimensional.*
2. *For $\lambda \neq 1$, $HH_0^\sigma(\mathcal{A}) \cong k$.*

Proof. We have $HH_0^\sigma(\mathcal{A}) = \{[a] \ : \ a = \sigma(a),\ [a_1a_2] = [\sigma(a_2)a_1]\}$. Hence for $\sigma = \sigma_\lambda$ with $\lambda \neq 1$, we need only consider P–B–W monomials A^n. Now,

$$cd[A^n] = [(A^2 + (d-c)A + B^*B)A^n] = [A^{n+2}] + (d-c)[A^{n+1}] + q^{2n}[\sigma(B)B^*A^n],$$
$$\Rightarrow [A^{n+2}] + (d-c)[A^{n+1}] - cd[A^n] = q^{2n}\lambda(q^4[A^{n+2}] + (d-c)q^2[A^{n+1}] - cd[A^n]),$$

so $f(n+2)[A^{n+2}] + (d-c)f(n+1)[A^{n+1}] - cdf(n)[A^n] = 0$, where $f(n) = \lambda^{-1} - q^{2n}$. Write $x_n = f(n)[A^n]$. Then we have

$$x_{n+2} + (d-c)x_{n+1} - cdx_n = 0 \quad \forall n \geq 0. \tag{18}$$

For $\lambda \notin q^{-2\mathbb{N}}$, we have $x_n = (c+d)^{-1}(\alpha c^n + \beta(-d)^n)$ with α, β given by:

$$\alpha = df(0)[1] + f(1)[A], \qquad \beta = cf(0)[1] - f(1)[A].$$

So for $\lambda \notin q^{-2\mathbb{N}}$, we have $HH_0^\sigma(\mathcal{A}) \cong k[1] \oplus k[A]$. There are three remaining cases we treat separately:

Case 1: $\sigma = \sigma_\lambda$, $\lambda = q^{-(2b+2)}$ ($b \geq 0$). Solving (18) requires some care. However, it is not difficult to show that:

1. $c \neq d$. $HH_0^\sigma(\mathcal{A}) \cong k[1] \oplus k[A^{b+1}]$. If $c \neq d$ then $[A], [A^{b+1}]$ also span.
2. $c = d$. If $\lambda = q^{-(4b+2)}$, then $HH_0^\sigma(\mathcal{A}) \cong k[1] \oplus k[A^{2b+1}]$.

For $\lambda = q^{-(4b+4)}$, $HH_0^\sigma(\mathcal{A}) \cong k[A] \oplus k[A^{2b+2}]$.

We give the proof of case 1 (case 2 is similar). For $\lambda = q^{-(2b+2)}$, $f(b+1) = 0$, hence $x_{b+1} = 0$. So (18) holds for $n \neq b, b \pm 1$. Hence for $n \geq b+2$ we have $x_n = (c+d)^{-1}(\alpha c^n + \beta(-d)^n)$ with

$$\alpha = c^{-(b+2)}[x_{b+3} + dx_{b+2}], \qquad \beta = (-d)^{-(b+2)}[cx_{b+2} - x_{b+3}].$$

Further, we have $x_{b+3} + (d-c)x_{b+2} = 0$, $x_{b+2} - cdx_b = 0$, so $x_{b+2} = cdx_b$, $x_{b+3} = cd(c-d)x_b$, hence $\alpha = c^{-b}dx_b$, $\beta = c(-d)^{-b}x_b$. Also, for $b \geq 1$ we have $(d-c)x_b - cdx_{b-1} = 0$. Finally, for $0 \leq n \leq b-2$ (provided $b \geq 2$) (18) holds, and solving this gives x_n for each $n \leq b$ in terms of x_b. We have, for each $b \geq 0$,

$$x_n = g(n-b-1)x_b, \quad \forall\, n \geq 0$$

where for $t \in \mathbb{Z}$, $g(t) = (c+d)^{-1}cd[c^t - (-d)^t]$. So for $cd = 0$, $[A^n] = 0$ for $n \neq 0, b+1$, while for $cd \neq 0$ each x_n, for $n \neq b+1$, is a nonzero multiple of x_b, and so of x_0. Since $f(n) \neq 0$ for $n \neq b+1$, we have $[A^n] = \rho_n[1]$, some $\rho_n \neq 0$, for each $n \neq b+1$. So for $b \geq 0$, $[1], [A^{b+1}]$, equivalently (for $b \geq 1$) $[A], [A^{b+1}]$, span $HH_0^\sigma(\mathcal{A})$. For nontriviality and linear independence, consider σ-twisted 0-cocycles τ_0, τ_{b+1}, defined (for $cd \neq 0$) on monomials $x = B^mA^n$ by

$$\tau_0(x) = \left\{\begin{matrix} \dfrac{g(n-b-1)}{f(n)} & : \ x = A^n, n \neq b+1 \\ 0 & : \ \text{otherwise} \end{matrix}\right\}, \qquad \tau_{b+1}(x) = \begin{cases} 1 & : \ x = A^{b+1} \\ 0 & : \ \text{otherwise.} \end{cases}$$

For $cd = 0$, define $\tau_0(1) = 1$, $\tau_0(x) = 0$ otherwise. Then for all $c \neq d$, $\tau_0(1) \neq 0$, $\tau_0(A^{b+1}) = 0$. So $HH_0^\sigma(\mathcal{A}) = k^2$, with basis $[1]$, $[A^{b+1}]$. We note the similarity of τ_0 with Noumi and Mimachi's $SU_q(2)$-invariant functional h (16), although the latter corresponds to the case $\lambda = q^2$.

Case 2: $\sigma = \sigma_\lambda$, $\lambda = 1$ ($\sigma = \mathrm{id}$). We have $x_0 = 0$, and:

1. $cd = 0, c - d \neq 0 : x_{n+1} = (c-d)^n x_1$ for all $n \geq 0$.
2. $cd \neq 0, c - d = 0 : x_{2n+1} = (cd)^n x_1, x_{2n+2} = 0$, for all $n \geq 0$.
3. $cd \neq 0, c - d \neq 0 :$ Then $x_{n+1} = g(n)x_1$, for some function g.

Also $[A^m B^n] = [\sigma(B^s)A^m B^{n-s}] = q^{2sm}[A^m B^n]$ for $0 \leq s \leq n$. So $[A^m B^n] = 0$ unless $m = 0$ or $n = 0$. Similarly for $[A^m B^{*n}]$. So for $\sigma = \mathrm{id}$, exactly as in [10],

$$HH_0^{\mathrm{id}}(\mathcal{A}) = H_0(\mathcal{A}, \mathcal{A}) \cong k[1] \oplus k[A] \oplus (\Sigma_{m\geq 1}^{\oplus} k[B^m]) \oplus (\Sigma_{m\geq 1}^{\oplus} k[B^{*m}]). \tag{19}$$

Case 3: $c = d$, $\sigma = \tau_\lambda$. Then $[A^{n+1}] = [A^n A] = [\sigma(A)A^n] = -[A^{n+1}]$. So $[A^{n+1}] = 0$ for $n \geq 0$. So for $\lambda \neq 1$, $HH_0^\sigma(\mathcal{A}) \cong k[1]$, and for $\lambda = 1$, $HH_0^\sigma(\mathcal{A})$ is given by (19), except that $[A] = 0$. □

4.2. $HH_1^\sigma(\mathcal{A})$

Proposition 4.2. *For $\sigma = \tau_\lambda$, if $\lambda \neq 1$ then $HH_1^\sigma(\mathcal{A}) = 0$. For $\lambda = 1$, $HH_1^\sigma(\mathcal{A})$ is countably infinite dimensional, spanned by $[(B^j, B)]$, $[(B^{*j}, B^*)]$, $j \geq 0$.*
For $\sigma = \sigma_\lambda$, and arbitrary c and d (with $c + d \neq 0$) we have

1. *For $\lambda = q^{-2}$ or $\lambda \notin q^{-2\mathbb{N}}$, $HH_1^\sigma(\mathcal{A}) \cong k[(1, A)]$.*
2. *For $\lambda = 1$ ($\sigma = \mathrm{id}$), $HH_1^\sigma(\mathcal{A})$ is countably infinite dimensional, spanned by $[(1, A)]$, $[(B^j, B)]$, $[(B^{*j}, B^*)]$ ($j \geq 0$).*
3. *For $cd = 0$, and $\lambda = q^{-(2b+4)}$ ($b \geq 0$), $HH_1^\sigma(\mathcal{A}) \cong k[(A^{b+1}, A)]$.*
4. *For $c - d = 0$, if $\lambda = q^{-(4b+4)}$, then $HH_1^\sigma(\mathcal{A}) \cong k[(1, A)] \oplus k[(A^{2b+1}, A)]$.*
   If $\lambda = q^{-(4b+6)}$, then $HH_1^\sigma(\mathcal{A}) \cong k[(A^{b+2}, A)]$.
5. *For $cd \neq 0$, $c - d \neq 0$, if $\lambda = q^{-4}$ then $HH_1^\sigma(\mathcal{A}) \cong k[(A, A)]$.*
   If $\lambda = q^{-(2b+6)}$, then $HH_1^\sigma(\mathcal{A}) \cong k[(1, A)] \oplus k[(A^{b+2}, A)]$

where for conciseness we denote by $[(x, y)]$ the class in $HH_1^\sigma(\mathcal{A})$ of $x \otimes y \in \mathcal{A}^{\otimes 2}$.

Proof. We have $d_1 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_1 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_0 \cong \mathcal{A}$ given by

$$d_1(a_1 \otimes e_A) = a_1.(A \otimes 1 - 1 \otimes A^o) = a_1 A - \sigma(A)a_1 = a_1 A - \mu A a_1,$$
$$d_1(a_2 \otimes e_{B^*}) = a_2.(B^* \otimes 1 - 1 \otimes B^{*o}) = a_2 B^* - \sigma(B^*)a_2 = a_2 B^* - \lambda^{-1} B^* a_2,$$
$$d_1(a_3 \otimes e_B) = a_3.(B \otimes 1 - 1 \otimes B^o) = a_3 B - \sigma(B)a_3 = a_3 B - \lambda B a_3.$$

($\mathcal{A}$ is a right $\mathcal{A}^e$-module via $a.(t_1 \otimes t_2{}^o) = \sigma(t_2)at_1$). So $(a_1, a_2, a_3) \in \ker(d_1) \Leftrightarrow$

$$(a_1 A - \mu A a_1) + (a_2 B^* - \lambda^{-1} B^* a_2) + (a_3 B - \lambda B a_3) = 0. \tag{20}$$

Suppose for fixed a_3 we have solutions (a_1', a_2', a_3), (a_1'', a_2'', a_3). Then $(a_1' - a_1'', a_2' - a_2'', 0)$ is a solution with $a_3 = 0$, and is moreover a solution of

$$(a_1 A - \mu A a_1) + (a_2 B^* - \lambda^{-1} B^* a_2) = 0. \tag{21}$$

So to calculate $\ker(d_1)/\mathrm{im}(d_2)$, we first show (Lemma 4.3) that (apart from one exceptional case) for any solution (a_1, a_2, a_3) there exists an element of $\mathrm{im}(d_2)$ with the same a_3. This reduces the problem to solving (21). Repeating this procedure, we show (Lemma 4.4) that except for two special cases any solution $(a_1, a_2, 0)$ is equivalent, modulo $\mathrm{im}(d_2)$, to a solution $(a_1', 0, 0)$, which reduces the problem to solving

$$a_1 A - \mu A a_1 = 0. \tag{22}$$

Suppose for any $a_3 = B^m A^k (m \in \mathbb{Z}, k \geq 0)$ we can either find a solution $a_1 = a_1(m, k)$, $a_2 = a_2(m, k)$ or show that none exists. Let $\mathcal{S} = \{(m, k) \in \mathbb{Z} \times \mathbb{N} :$ (20) has a solution with $a_3 = B^m A^k\}$. Then any solution of (20) is of the form

$$a_3 = \sum_{\mathcal{S}} \alpha_{m,k} B^m A^k, \quad a_2 = \sum_{\mathcal{S}} \alpha_{m,k} a_2(m, k) + a_2{}', \quad a_1 = \sum_{\mathcal{S}} \alpha_{m,k} a_1(m, k) + a_1{}' + a_1{}''$$

for some $\alpha_{m,k} \in k$, where $(a_1{}', a_2{}')$, $a_1{}''$ are solutions of (21) and (22).

We have $d_2 : \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_2 \to \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{M}_1$ given by

$$\begin{aligned} &d_2[b_1 \otimes e_A \wedge e_{B^*} + b_2 \otimes e_A \wedge e_B + b_3 \otimes \vartheta_S^{(1)} + b_4 \otimes \vartheta_T^{(1)}] \\ &\quad = [(\lambda^{-1} B^* b_1 - q^2 b_1 B^*) + (q^2 \lambda B b_2 - b_2 B) - q(q^2(b_3 A + \mu A b_3) + (d - c) b_3) \\ &\qquad - q^{-1}(b_4 A + \mu A b_4 + (d - c) b_4)] \otimes e_A \qquad (23) \\ &\qquad + [(b_1 A - q^2 \mu A b_1) - q^{-1}(b_3 B + \lambda B b_4)] \otimes e_{B^*} \qquad (24) \\ &\qquad + [(q^2 b_2 A - \mu A b_2) - q^{-1}(\lambda^{-1} B^* b_3 + b_4 B^*)] \otimes e_B. \qquad (25) \end{aligned}$$

Lemma 4.3. *Given $(a_1, a_2, a_3) \in \ker(d_1)/\mathrm{im}(d_2)$, we can take $a_3 = 0$ unless $\lambda = 1$, in which case the space of (equivalence classes of) solutions with $a_3 \neq 0$ is spanned (as a k-vector space) by $\{a_1 = 0 = a_2,\ a_3 = B^j,\ j \geq 0\}$.*

Proof. To solve (20) with $a_3 = B^{*j+1} A^k$, take $b_3 = -q\lambda B^{*j} A^k$, $b_1 = 0 = b_2 = b_4$ in (25). To solve (20) with $a_3 = B^j A^{k+1}$, take $b_2 = B^j A^k$, all other b_i zero in (25). This leaves the case of solving (20) with $a_3 = B^j$. Take $b_1 = 0$,

$$b_2 = B^j[q^{-2j}(1 + x^2)A + (d - c)(1 + x)], \quad b_3 = q\lambda(\mu - x^2)B^{j+1}, \quad b_4 = q(\mu - x)B^{j+1}$$

where $x = q^{2j+2}$, giving $a_3 = (x - \mu)cdB^j$. So for $cd \neq 0$ we're done. For $cd = 0$, it is clear there is no solution to (20) with $a_3 = B^j$ unless $\lambda = 1$, in which case $a_2 = a_1 = 0$ gives a solution. □

So we have reduced solving (20) modulo $\mathrm{im}(d_2)$ to solving (21). In the same way, it is straightforward to show that:

Lemma 4.4. *Any solution of (20) with $a_3 = 0$ is equivalent, modulo $\mathrm{im}(d_2)$, either to a solution with $a_3 = 0 = a_2$, or to one of the special cases:*

1. *$\lambda = 1$, $\mu = \pm 1$, $a_1 = 0 = a_3$, $a_2 = (B^*)^j$, $j \geq 0$.*
2. *$\lambda = 1$, $\mu = \pm 1$, $a_3 = 0$, $a_1 = B^j[f(2j + 2)q^{-2j}A + (d - c)f(j + 1)]$, $a_2 = (\mu - q^{2j})B^{j+1}$, $j \geq 0$, which is equivalent to $a_1 = 0 = a_2$, $a_3 = B^j$. Here $f(n) = \lambda^{-1} - q^{2n}$ as before.*

Finally we need to solve (22). For $\mu = -1$, the only solution is $a_1 = 0$.

Lemma 4.5. *For $\mu = 1$, $\mathcal{V} = \{(a_1, 0, 0) \in \ker(d_1)/\mathrm{im}(d_2)\}$ is spanned by:*

1. *If $\lambda \notin \{q^{-(2b+4)}\}_{b \geq 0}$, then $(1, 0, 0)$ spans $\mathcal{V}$.*
2. *$cd = 0$, $\lambda = q^{-(2b+4)}$. Then $(A^{b+1}, 0, 0)$ spans $\mathcal{V}$.*
3. *$c - d = 0$. For $\lambda = q^{-(4b+4)}$, $(A^{2b+1}, 0, 0)$, $(1, 0, 0)$ span $\mathcal{V}$.*
   For $\lambda = q^{-(4b+6)}$, $(A^{2b+2}, 0, 0)$ spans $\mathcal{V}$.
4. *$cd \neq 0$, $c - d \neq 0$. For $\lambda = q^{-4}$, $(A, 1, 1)$ spans $\mathcal{V}$.*
   For $\lambda = q^{-(2b+6)}$, $(A^{b+2}, 0, 0)$, $(1, 0, 0)$ span $\mathcal{V}$.

Proof. For $\mu = 1$, the space of solutions of (22) is spanned by $\{a_1 = A^j,\ j \geq 0\}$. These solutions are not linearly independent. Take $b_1 = BA^j$, $b_2 = 0 = b_3 = b_4$ in (23)–(25), giving $a_2 = 0 = a_3$, $a_1 = cdf(j + 1)A^j + (c - d)f(j + 2)A^{j+1} - f(j + 3)A^{j+2}$. Let $y_n = f(n + 1)[A^n \otimes e_A] \in \ker(d_1)/\mathrm{im}(d_2)$. So we have

$$y_{n+2} + (d - c)y_{n+1} - cdy_n = 0 \quad \forall\, n \geq 0. \qquad (26)$$

This is the same recursion relation as (18). In addition, taking $b_1 = 0 = b_2$, $b_3 = q$, $b_4 = -q\lambda^{-1}$ in (23)–(25), gives $a_2 = 0 = a_3$, $a_1 = 2f(2)A + (d - c)f(1)$, hence $2y_1 = (c - d)y_0$. Solving (26) in the same manner as for (18) in the proof of Proposition 4.1, together with this additional constraint gives the result. □

Given $a_1 \otimes e_A + a_2 \otimes e_{B^*} + a_3 \otimes e_B \in \ker(d_1)/\mathrm{im}(d_2)$ we manufacture a twisted Hochschild 1-cycle using (12). Collecting the results of Lemmas 4.3–4.5 gives the description of $\ker(d_1)/\mathrm{im}(d_2)$ appearing in the statement of Proposition 4.2. This completes the proof of Proposition 4.2.

4.3. $HH_n^\sigma(\mathcal{A})$, $n \geq 2$

Theorem 4.6. *For arbitrary c and d (with $c + d \neq 0$), we find that:*

1. *For $\sigma = \sigma_\lambda$, $\lambda = q^{-(2b+2)}$, some $b \geq 0$, then $HH_2^\sigma(\mathcal{A}) \cong k$. These automorphisms are precisely the positive powers of the modular automorphism σ_{mod} (17) induced from the Haar state on quantum $SU(2)$.*
2. *For all other σ_λ, τ_λ, $HH_2^\sigma(\mathcal{A}) = 0$.*

The proof proceeds in the same manner as Proposition 4.2, using (11) and (12). We omit the details. For $\lambda = q^{-(2b+2)}$, $HH_2^\sigma(\mathcal{A}) \cong k[\omega_2]$, where ω_2 is the twisted Hochschild 2-cycle:

$$\begin{aligned}\omega_2 = {} & 2[(A^{b+1}, B, B^*) - (A^{b+1}, B^*, B) + 2(A^b B, B^*, A) - 2q^{-2}(A^b B, A, B^*)] \\ & + 2(q^4 - 1)(A^{b+1}, A, A) + (1 - q^{-2})cd(c-d)(A^b, 1, 1) \\ & + (c-d)[(A^b, B^*, B) - q^{-2}(A^b, B, B^*) + (1-q^2)(A^b, A, A)]. \end{aligned} \tag{27}$$

Finally, all the higher twisted Hochschild homology groups vanish:

Proposition 4.7. *We have $HH_n^\sigma(\mathcal{A}) = 0$ for all $n \geq 3$ for any $\sigma = \sigma_\lambda, \tau_\lambda$.*

We prove this in the case $n = 3$:

Theorem 4.8. *$HH_3^\sigma(\mathcal{A}) = 0$ for all automorphisms $\sigma = \sigma_\lambda, \tau_\lambda$.*

Proof. We have

$$d_3[a_1 \otimes (e_A \wedge \vartheta_S^{(1)}) + a_2 \otimes (e_A \wedge \vartheta_T^{(1)}) + a_3 \otimes (e_{B^*} \wedge \vartheta_S^{(1)}) + a_4 \otimes (e_B \wedge \vartheta_T^{(1)})]$$

$$= [q^{-3}a_1 B + q^{-1}\lambda B a_2 - q^{-1}(a_3 A + q^2 \mu A a_3 - (c-d)a_3)] \otimes (e_A \wedge e_{B^*}) \tag{28}$$

$$+ [q^{-3}\lambda^{-1} B^* a_1 + q^{-1} a_2 B^* - q^{-1}(q^2 a_4 A + \mu A a_4 - (c-d)a_4)] \otimes (e_A \wedge e_B) \tag{29}$$

$$+ [(a_1 A - \mu A a_1) + a_3 B^* - \lambda B a_4] \otimes \vartheta_S^{(1)} \tag{30}$$

$$+ [(a_2 A - \mu A a_2) - \lambda^{-1} B^* a_3 + a_4 B] \otimes \vartheta_T^{(1)} \tag{31}$$

and

$$\begin{aligned} & d_4[b_1 \otimes (e_A \wedge e_{B^*} \wedge \vartheta_S^{(1)}) + b_2 \otimes (e_A \wedge e_B \wedge \vartheta_T^{(1)}) + b_3 \otimes \vartheta_S^{(2)} + b_4 \otimes \vartheta_T^{(2)}] \\ & = [-q^2 b_1 B^* + q^2 \lambda B b_2 - q(q^2 b_3 A + \mu q^2 A b_3 - (c-d)b_3)] \otimes (e_A \wedge \vartheta_S^{(1)}) \\ & \quad + [\lambda^{-1} B^* b_1 - b_2 B - q^{-1}(b_4 A + \mu A b_4 - (c-d)b_4)] \otimes (e_A \wedge \vartheta_T^{(1)}) \\ & \quad + [(b_1 A - q^2 \mu A b_1) - q^{-1} b_3 B - q^{-1}\lambda B b_4] \otimes (e_{B^*} \wedge \vartheta_S^{(1)}) \\ & \quad + [(q^2 b_2 A - \mu A b_2) - q^{-1}\lambda^{-1} B^* b_3 - q^{-1} b_4 B^*] \otimes (e_B \wedge \vartheta_T^{(1)}). \end{aligned} \tag{32}$$

Finding $\ker(d_3)$ corresponds to finding all solutions (a_1, a_2, a_3, a_4) to the four Eqs. (28)–(31). Our strategy is the same as for Proposition 4.2. Suppose for fixed a_4 we find solutions (a_1, a_2, a_3, a_4), (a_1', a_2', a_3', a_4). Then $(a_1 - a_1', a_2 - a_2', a_3 - a_3', 0)$ is a solution with $a_4 = 0$. So to calculate $\ker(d_3)/\mathrm{im}(d_4)$, we first show (Lemma 4.9) that for any solution (a_1, a_2, a_3, a_4) there exists an element of $\mathrm{im}(d_4)$ with the same a_4. So we need only look for solutions with $a_4 = 0$.

We repeat this procedure for a_3 (Lemma 4.10), showing that $\ker(d_3)/\mathrm{im}(d_4)$ is spanned by (equivalence classes of) solutions with $a_3 = 0 = a_4$. Finally we show (Lemmas 4.11 and 4.12 that any such solution belongs to $\mathrm{im}(d_4)$.

Lemma 4.9. *Any solution (a_1, a_2, a_3, a_4) of (28)–(31) is equivalent, modulo $\mathrm{im}(d_4)$, to a solution with $a_4 = 0$.*

Proof. We start by solving for given a_4. It is enough just to consider monomials. For $a_4 = (B^*)^{j+1}A^k$, take $b_3 = -q\lambda(B^*)^j A^k$, $b_2 = 0 = b_4$ in (32). To solve for $a_4 = B^j A^{k+1}$, take $b_2 = (q^2 - \mu q^{-2j})^{-1}B^j A^k$, $b_3 = 0 = b_4$. Then $q^2 b_2 A - \mu A b_2 = B^j A^{k+1}$. So we are left with only the case $a_4 = B^j$. Take $b_2 = (q^2 - \mu q^{-2j})^{-1}B^j[\alpha_0 + \alpha_1 A]$, $b_3 = -qB^{j+1}$, $b_4 = 0$. Then $-q^{-1}b_4 B^* - q^{-1}\lambda^{-1}B^* b_3 = B^j[cd + q^2(c-d)A + q^4A^2]$ and $q^2 b_2 A - \mu A b_2 = B^j[\alpha_0 A + \alpha_1 A^2]$. Taking $\alpha_0 = q^2(d-c)$, $\alpha_1 = -q^4$, we see that provided $cd \neq 0$, we can find solutions with $a_4 = B^j$ for any $j \geq 0$. If $cd = 0$, then we see from (30) and (31) that $a_4 = B^j$ cannot be in $\ker(d_3)$. □

In the same way, it is straightforward to show that:

Lemma 4.10. *Any solution* (a_1, a_2, a_3, a_4) *of* (28)–(31) *with* $a_4 = 0$ *is equivalent, modulo* $\mathrm{im}(d_4)$*, to a solution with* $a_3 = a_4 = 0$.

So we need only consider $a_1, a_2 \neq 0$. From (30) and (31), we have

$$a_1 A = \mu A a_1, \quad a_2 A = \mu A a_2. \tag{33}$$

Lemma 4.11. *For* $\mu = -1$*, the only solution to* (33) *is* $a_1 = 0 = a_2$.

Hence for $\mu = -1$, $\ker(d_3) = \mathrm{im}(d_4)$, thus proving Theorem 4.8 in this case.

For $\mu = 1$, (28) and (29) give $a_1 B + q^2\lambda B a_2 = 0$, $B^* a_1 + q^2 \lambda a_2 B^* = 0$ (it is straightforward to show that these two conditions are equivalent). So for $\mu = 1$, $\ker(d_3)/\mathrm{im}(d_4)$ is spanned by (the equivalence classes of) the solutions

$$a_1 = -\lambda q^{2j+2}A^j, \quad a_2 = A^j, \quad a_3 = 0 = a_4 \quad (j \geq 0). \tag{34}$$

Lemma 4.12. *The solutions* (34) *all belong to* $\mathrm{im}(d_4)$.

Proof. In the case $cd \neq 0$, $c - d \neq 0$, taking

$$\begin{aligned} &b_1 = 4\alpha_1 B A^j, \qquad b_2 = 4q^{2j}(\alpha_1\lambda^{-1} - \gamma)B^* A^j, \qquad \gamma = 4(c+d)^{-2} \\ &b_3 = \lambda\gamma q^{2j+1}A^j[2q^2 A - (c-d)], \qquad b_4 = -\gamma q A^j[2A - (c-d)] \end{aligned}$$

some $\alpha_1 \neq \lambda\gamma$, gives (34). The other two cases ($cd = 0$, $c = d$) are similar. □

This completes the proof of Theorem 4.8. □

5. Twisted cyclic homology of the Podleś spheres

For an algebra $\mathcal{A}$ and automorphism σ, twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ arises as in [9] from the cyclic module C^σ, with objects $\{C_n^\sigma\}_{n\geq 0}$ (2) defined by $C_n^\sigma = \mathcal{A}^{\otimes(n+1)}/(\mathrm{id} - \sigma^{\otimes(n+1)})$. The face, degeneracy and cyclic operators were given explicitly in [5]. Twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ is the total homology of Connes' mixed (b, B)-bicomplex corresponding to the cyclic module C^σ:

$$\begin{array}{ccccccc}
\downarrow b_4 & & \downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 \\
C_3^\sigma & \xleftarrow{B_2} & C_2^\sigma & \xleftarrow{B_1} & C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma \\
\downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 & & \\
C_2^\sigma & \xleftarrow{B_1} & C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma & & \\
\downarrow b_2 & & \downarrow b_1 & & & & \\
C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma & & & & \\
\downarrow b_1 & & & & & & \\
C_0^\sigma & & & & & &
\end{array} \tag{35}$$

The maps b_n coincide with the twisted Hochschild boundary maps b_σ (3). We will drop the suffixes and write b_n, b_σ as b. In lowest degrees, the maps B_n are:

$$B_0[a_0] = [(1, a_0)] + [(\sigma(a_0), 1)] = [(1, a_0)] + [(a_0, 1)],$$
$$B_1[(a_0, a_1)] = [(1, a_0, a_1)] - [(\sigma(a_1), 1, a_0)] - [(1, \sigma(a_1), a_0)] + [(a_0, 1, a_1)].$$

For any $a \in \mathcal{A}$, $b(a, 1, 1) = (a, 1)$, so $[(a, 1)] = 0$ in $HH_1^\sigma(\mathcal{A})$. So the induced map $B_0 : HH_0^\sigma(\mathcal{A}) \to HH_1^\sigma(\mathcal{A})$ satisfies $B_0[a] = [(1, a)]$. For $t \in \mathcal{A}$, with $\sigma(t) = \alpha t$, some $\alpha \in k$, then

$$b\left(\sum_{j=0}^{m} \alpha^j (t^j, t^{m-j}, t) - (t^{m+1}, 1, 1)\right) = \left(\sum_{j=0}^{m} \alpha^j\right) (t^m, t) - (1, t^{m+1}). \tag{36}$$

If $\alpha = 1$, then $B_0[t^{m+1}] = [(1, t^{m+1})] = (m + 1)[(t^m, t)] \in HH_1^\sigma(\mathcal{A})$.

Taking $\mathcal{A} = \mathcal{A}(c, d)$, we calculate total homology of the mixed complex (35) via a spectral sequence. The first step (vertical homology of the columns) gives:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{0} & HH_2^\sigma(\mathcal{A}) & \xleftarrow{B_1} & HH_1^\sigma(\mathcal{A}) & \xleftarrow{B_0} & HH_0^\sigma(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow & & \\
HH_2^\sigma(\mathcal{A}) & \xleftarrow{B_1} & HH_1^\sigma(\mathcal{A}) & \xleftarrow{B_0} & HH_0^\sigma(\mathcal{A}) & & \\
\downarrow & & \downarrow & & & & \\
HH_1^\sigma(\mathcal{A}) & \xleftarrow{B_0} & HH_0^\sigma(\mathcal{A}) & & & & \\
\downarrow & & & & & & \\
HH_0^\sigma(\mathcal{A}) & & & & & &
\end{array} \tag{37}$$

since for every σ we have $HH_n^\sigma(\mathcal{A}) = 0$ for $n \geq 3$. We find that:

Proposition 5.1. *For $\lambda \notin q^{-2\mathbb{N}}$, $\sigma = \sigma_\lambda$, $HC_{2n}^\sigma(\mathcal{A}) = k[1] \oplus k[A]$, $HC_{2n+1}^\sigma(\mathcal{A}) = 0$. For $\sigma = \tau_\lambda$ with $\lambda \neq 1$, $HC_n^\sigma(\mathcal{A}) = 0$ for $n \geq 1$.*

Proof. In both cases $HH_0^\sigma(\mathcal{A}) = k[1] \oplus k[A]$ (with $[A] = 0$ for $\sigma = \tau_\lambda$), $HH_n^\sigma(\mathcal{A}) = 0$ for $n \geq 1$, (37) stabilizes immediately, and the result follows. □

Proposition 5.2. *For $\lambda = 1$, $\mu = \pm 1$, then just as in* [10] *we have*

$$HC_0^\sigma(\mathcal{A}) = k[1] \oplus k[A] \oplus \left(\sum_{m>0}^{\oplus} k[B^m]\right) \oplus \left(\sum_{m>0}^{\oplus} k[B^{*m}]\right)$$
$$HC_{2n+1}^\sigma(\mathcal{A}) = 0, \quad HC_{2n+2}^\sigma(\mathcal{A}) = k[1] \oplus k[A], \quad \text{with } [A] = 0 \text{ for } \mu = -1.$$

Proof. We have $B_0[1] = 0$, $B_0[A] = [(1, A)] = 0$, while $B_0[B^{m+1}] = [(1, B^{m+1})] = (m + 1)[(B^m, B)]$ by (36), and in the same way $B_0[B^{*m+1}] = (m + 1)[(B^{*m}, B^*)]$. So $\ker(B_0) = k[1] \oplus k[A]$, and $HH_1^\sigma(\mathcal{A}) = \mathrm{im}(B_0)$. Hence the spectral sequence stabilizes at the second page with all further maps being zero. □

Proposition 5.3. *For $\sigma = \sigma_\lambda$, $\lambda = q^{-(2b+2)}$, then $HC_{2n+1}^\sigma(\mathcal{A}) = 0$, and:*

1. $\lambda = q^{-2}$. $HC_{2n+2}^\sigma(\mathcal{A}) = k[1] \oplus k[\omega_2]$.
2. $\lambda = q^{-4}$. *For $c = d$, $HC_{2n+2}^\sigma(\mathcal{A}) = k[\omega_2]$, else $HC_{2n+2}^\sigma(\mathcal{A}) = k[1] \oplus k[\omega_2]$.*
3. $\lambda = q^{-(4b+6)}$. *For $cd = 0$ or $c = d$, $HC_{2n+2}^\sigma(\mathcal{A}) = k[1] \oplus k[\omega_2]$, otherwise $HC_{2n+2}^\sigma(\mathcal{A}) = k[\omega_2]$.*
4. $\lambda = q^{-(4b+8)}$. *For $cd = 0$, $HC_{2n+2}^\sigma(\mathcal{A}) = k[1] \oplus k[\omega_2]$, otherwise $HC_{2n+2}^\sigma(\mathcal{A}) = k[\omega_2]$.*

Proof. We prove case 3, the others are completely analogous. For $cd = 0$ or $c = d$, $HH_0^\sigma(\mathcal{A}) = k[1] \oplus k[A^{2b+3}]$, and $HH_1^\sigma(\mathcal{A}) = k[(A^{b+2}), A]$. We have $B_0[1] = [(1, 1)] = 0$, $B_0[A^{2b+3}] = [(1, A^{2b+3})] = (2b+3)[(A^{b+2}, A)]$. So $\ker(B_0) = k[1]$, $\operatorname{im}(B_0) = HH_1^\sigma(\mathcal{A})$. Then the spectral sequence (37) stabilizes at page two:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & k[\omega_2] & \longleftarrow & 0 & \longleftarrow & k[1] \\
\downarrow & & \downarrow & & \downarrow & & \\
k[\omega_2] & \longleftarrow & 0 & \longleftarrow & k[1] & & \\
\downarrow & & \downarrow & & & & \\
0 & \longleftarrow & k[1] & & & & \\
\downarrow & & & & & & \\
HH_0^\sigma(\mathcal{A}) & & & & & &
\end{array}$$

with all further maps being zero. For $cd \neq 0$ and $c \neq d$, then $HH_0^\sigma(\mathcal{A}) = k[1] \oplus k[A^{2b+3}] = k[A] \oplus k[A^{2b+3}]$, and $HH_1^\sigma(\mathcal{A}) = k[(1, A)] \oplus k[(A^{b+2}), A]$. Then $B_0[A] = [(1, A)]$, hence $\ker(B_0) = 0$, $\operatorname{im}(B_0) = HH_1^\sigma(\mathcal{A})$. □

6. The standard Podleś quantum sphere

We specialize our results to the standard quantum sphere $\mathcal{A}(S_q^2)$, which as described in Section 3 naturally embeds as a *-subalgebra of $\mathcal{A}(SU_q(2))$. We recall that Schmüdgen and Wagner [17] defined a twisted cyclic 2-cocycle τ over $\mathcal{A}(S_q^2)$ as follows. For $a_0, a_1, a_2 \in \mathcal{A}(S_q^2)$, define

$$\tau(a_0, a_1, a_2) = h(a_0[(a_1 \triangleleft F)(a_2 \triangleleft E) - q^2(a_1 \triangleleft E)(a_2 \triangleleft F)]) \tag{38}$$

where $\triangleleft$ is the right action of $U_q(su(2))$ (8). As shown in [17], the mappings $\mathcal{A}(S_q^2) \to \mathcal{A}(SU_q(2))$ given by $x \mapsto x \triangleleft E$, $x \mapsto x \triangleleft F$ are derivations. Here h denotes the Haar state on $\mathcal{A}(SU_q(2))$, which restricts to $\mathcal{A}(S_q^2)$ as

$$h(A^r B^s) = 0 = h(A^r (B^*)^s) \quad s > 0, \qquad h(A^r) = (1 - q^2)(1 - q^{2r+2})^{-1}.$$

Schmüdgen and Wagner proved:

Proposition 6.1 (*[17], Theorem 4.5*). *τ is a nontrivial σ-twisted cyclic 2-cocycle on $\mathcal{A}(S_q^2)$, with σ the automorphism given by $\sigma(x) = K^{-2} \triangleright x$. Further, τ is $U_q(su(2))$-invariant and coincides with the volume form of the distinguished covariant 2-dimensional first order differential calculus found by Podleś* [14].

Schmüdgen and Wagner also constructed a $U_q(su(2))$-equivariant Dirac operator, unitarily equivalent to those previously found by Bibikov and Kulish [1] and Dabrowski and Sitarz [3], which they used to give a representation of the Podleś calculus by bounded commutators.

Explicitly, $\sigma(B) = q^2 B$, $\sigma(B^*) = q^{-2} B^*$. So in (14), $\lambda = q^2$. From Proposition 4.2 and Theorem 4.6 we have $HH_n^\sigma(\mathcal{A}) = 0$ for $n \geq 1$ for this σ, i.e. this twisted cocycle does not correspond to the "no dimension drop" case. By Proposition 5.1, we have $HC_{2n}^\sigma(\mathcal{A}) = \mathbb{C}[1] \oplus \mathbb{C}[A]$, $HC_{2n+1}^\sigma(\mathcal{A}) = 0$ for all $n \geq 0$. The σ-twisted cyclic 0-cocycles τ_0, h_A dual to $[1]$, $[A]$ are defined on Poincaré–Birkhoff–Witt monomials x (6) by $\tau_0(1) = 1$, $\tau_0(x) = 0$ for $x \neq 1$, and

$$\begin{aligned}
&h_A(A^r B^s) = 0 = h_A(A^r (B^*)^s) \quad s > 0 \\
&h_A(1) = 0, \quad h_A(A^{r+1}) = (1 - q^4)(1 - q^{2r+4})^{-1}.
\end{aligned}$$

The Haar state h (restricted to $\mathcal{A}(S_q^2)$) is given by $h = \tau_0 + (1 + q^2)^{-1} h_A$. By cohomology calculations completely dual to our previous homology calculations, we have $HC_\sigma^{2n}(\mathcal{A}) \cong \mathbb{C}[S^n \tau_0] \oplus \mathbb{C}[S^n h_A]$, $HC_\sigma^{2n+1}(\mathcal{A}) = 0$, where S is Connes' periodicity operator. We can now identify the class of τ in $HC_\sigma^2(\mathcal{A})$:

Theorem 6.2. *We have* $[\tau] = \beta[Sh_A] \in HC^2_\sigma(\mathcal{A})$*, for some nonzero* β.

Proof. We have $HC^2_\sigma(\mathcal{A}) \cong \mathbb{C}^2$, generated by $[S\tau_0]$, $[Sh_A]$, where $S\phi(a_0, a_1, a_2) = \phi(a_0a_1a_2)$ for any $\phi \in HC^0_\sigma(\mathcal{A})$. Recall from [17] the element

$$\eta = (B^*, A, B) + q^2(B, B^*, A) + q^2(A, B, B^*) - q^{-2}(B^*, B, A) \\ - q^{-2}(A, B^*, B) - (B, A, B^*) + (q^6 - q^{-2})(A, A, A).$$

Now, $\tau(\eta) = -1$, and it was shown in [17] that $[\tau]$ is nontrivial in $HC^2_\sigma(\mathcal{A})$. So there are scalars α, β, not both zero, such that $[\tau] = \alpha[S\tau_0] + \beta[Sh_A]$. Now, $\tau(1, 1, 1) = 0 = Sh_A(1, 1, 1)$, whereas $S\tau_0(1, 1, 1) = \tau_0(1) = 1$. Hence $\alpha = 0$. Since $S\eta = (q^4 - q^{-2})A^2$, we have $Sh_A(\eta) = h_A(S\eta) = (q^4 - q^{-2})h_A(A^2) = q^2 - q^{-2}$. If η was a twisted 2-cycle we could deduce that $\beta = (q^{-2} - q^2)^{-1}$. Since $b_\sigma(\eta) = 2(q^4 - q^{-2})(A, A) \neq 0$, this need not hold. We could calculate β by finding $[\mathbf{a}] \in HC^\sigma_2(\mathcal{A})$ such that $[S\mathbf{a}] = [A] \in HC^\sigma_0(\mathcal{A})$ (note that $(1 - q^{2s+4})[A^{s+1}] = (1 - q^4)[A]$ for $s \geq 0$). Then $\tau(\mathbf{a}) = \beta Sh_A(\mathbf{a}) = \beta h_A(S\mathbf{a}) = \beta h_A(A) = \beta$. However finding such an $\mathbf{a}$ explicitly has not been possible. □

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